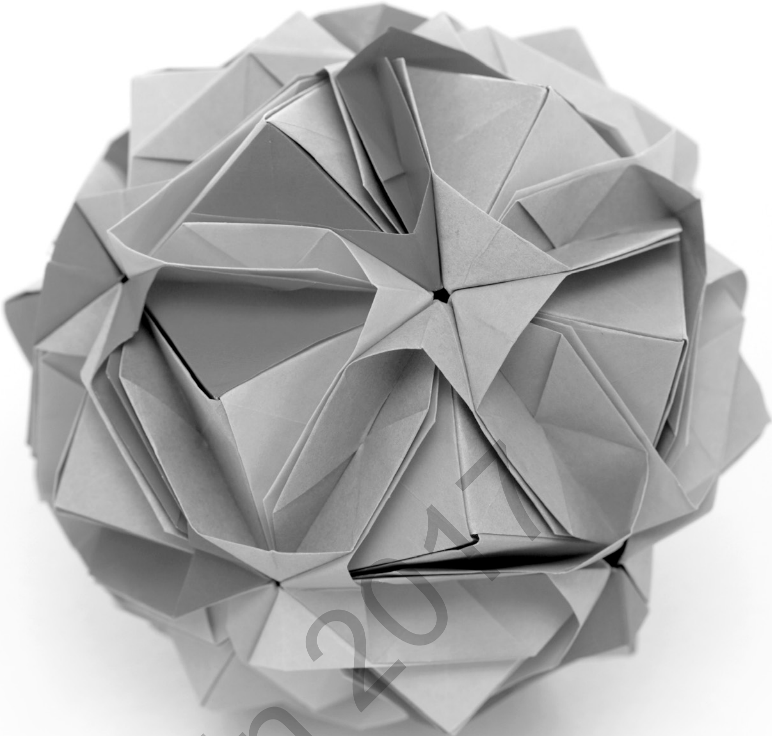

CHAPTER THREE

TEACH UP

MAKING SENSE OF RIGOROUS MATHEMATICAL CONTENT

The second foundational key to effective differentiation is to know the mathematics content in depth. Differentiation will be purposeful and effective only when the mathematics standards are analyzed and form the basis of instruction and activities. In this chapter, you will find the following:

Mathematics Makes Sense
Themes and Big Ideas in Mathematics
Teaching Up
What Learning Mathematics With Understanding Looks Like
Frequently Asked Questions
Keepsakes and Plans

If I were to ask you how to complete this sentence, what would you say: "The most basic idea in the learning of mathematics is . . . "? What did you say? Patterns? Number sense? Calculations? Perhaps you went a different way and said applications. Or modeling. As many times as I ask teachers this question, I receive a wide variety of answers. I have never heard anyone complete the statement in accordance with the original quote, however. According to John Van de Walle (2007), "The most basic idea in the learning of mathematics is . . . mathematics makes sense." Truthfully, I didn't think of that answer the first time I saw this quote either! The more I ponder it and the more I work with teachers in constructing powerful mathematics lessons, the more I realize that it should be every teacher's mantra. I am just wondering—are there some of you right now thinking that mathematics *doesn't* make sense? Do you have students who would not believe the statement that mathematics makes sense?

This is a chapter on mathematical content and trying to make sense of it, which is foundational to differentiation. If we design differentiated tasks before we have clear conceptual understanding, knowledge, and skills of the content we are teaching, we are probably dooming ourselves to a lot of extra planning and sometimes frustrating classroom experiences, with little growth or achievement to show as a result. To paraphrase something Carol Ann Tomlinson once said (personal communication), "If we are somewhat foggy in what we are teaching, and then differentiate, we end up with differentiated fog." This is not our goal.

MATHEMATICS MAKES SENSE

Our brains are sense-making machines. In fact, our brains naturally seek patterns and meaning making in order to store to long-term memory (Sousa, 2015). We now know that in order for an idea or concept to be stored in long-term memory, it needs to make sense and be relevant to the learner. Unfortunately, we often do not teach math as if it is a sense-making subject. We tend to teach skills, or problem types, and then practice, practice, practice . . . until we reach the next skill to be taught. I would not be surprised if mathematics was the instigator of the phrase "drill and kill." However, if we can begin to view mathematics as sense-making, we can break this pattern, both for ourselves as teachers and for our students. Most teachers I meet have never been taught to understand mathematics, only how to do it—even as mathematics majors in college. It makes sense, then, that many teachers struggle to make mathematics understandable for their students, especially if mathematics just naturally made sense to us throughout our schooling.

We have learned from cognitive science that the human brain is not well designed for memorizing data. It is most efficient and effective when it works with patterns, connections, meaning, and significance—with personal meaning being the most important (Sousa, 2015). Without the necessary time to make sense of learning, students naturally resort to memorization. Thus, the most effective way for students to learn mathematics is to prioritize understanding rather than memorization. It is our job to provide lessons that can make that happen.

How do we begin to make sense of mathematics? The first step is to clarify the big ideas that are the foundation for the topic(s) being taught in the unit. Sometimes these essential understandings are embedded in the standard we are addressing, and sometimes they can be seen in some of the exploration tasks in a resource, but it is almost always up to the teacher (or a collaborative team if you are working in one) to determine.

THEMES AND BIG IDEAS IN MATHEMATICS

There are some big ideas in mathematics that are true throughout mathematics, for every grade level and every course. Figure 3.1 provides a few of these concepts and understandings.

This figure is just the beginning of thinking in terms of broad conceptual understanding in mathematics. These understandings span units and grade levels as you can see. They are as true in kindergarten as they are in calculus. Let's take another step. For any mathematical unit based on a group of standards to be taught, the content can be divided into what students will come to know, understand, and be able to do. In the differentiation literature, this is referred to as KUDs (Tomlinson & Imbeau, 2014; Tomlinson & Moon, 2013).

The **K**now in KUD are about facts that can be memorized. Our mathematics content is filled with Knows; math facts, vocabulary, formulas, and steps to a procedure (such as how to plot points) all fall under the Know category. If you can look it up, it is probably a Know. On the other hand, **U**nderstandings are conceptual. They are big ideas and have many layers. Understandings connect the content across units, as well as connect mathematical content to other subjects. Understandings remain important and true over years,

Consider It!

- What is the difference between knowing, understanding, and doing mathematics?
- What does it look like when students exhibit understanding of mathematics? How is it different from students who know *how to* do mathematics but don't *understand* mathematics?

FIGURE 3.1

GENERAL CONCEPTS AND UNDERSTANDINGS IN MATHEMATICS

Concept	Understandings
Mathematical Operations and Properties	• Each operation in mathematics has meanings that make sense of situations, and the essential meaning of each operation remains true in every context and number system. • Every mathematical operation has specific properties that apply to it, and these properties are the basis for how these operations can and cannot be used. • The properties of operations provide the reasoning for mathematical explanations.
Number Sense and Estimation	• Developing mental mathematics strategies that reason about numbers, quantities, and the operations with numbers provide flexibility and confidence in working with numbers. • Estimation allows for establishing the reasonableness of an answer.
Units and "Unitizing"	• Determining the "base entity" in a given context or problem (e.g., apples, balloons, rate or ratio, variable, term, and possibly a function) allows you to make sense of the problem, plan a solution path, and make comparisons. • Units in measurement describe what is being measured, and what is being measured has a specific type of unit.
Equality	• An equal sign is a statement that two quantities are equivalent. That equivalency must be maintained throughout any mathematical manipulations or operations.
Shape and Geometry	• Shapes and their properties describe our physical world. • Shapes are categorized and grouped according to their properties. • Relationships among shapes can be described in many ways, including algebraically.
Modeling and Representation	• There are many different representations for a given problem or situation, and each representation can highlight or reveal different aspects of the problem. • Mathematical models represent real-world contexts and provide for connections, comparisons, and predictions.

as seen in Figure 3.1, and in fact, it is powerful if the same understandings are used over time to clearly show students that all mathematics topics are connected. Finally, the **D**o is what you expect students to be able to do if they truly Know and Understand. Be careful not to list specific task activities (e.g., "make a poster to show steps for Questions 3 through 8") in the Do category. You are looking for the mathematics within any task that indicate knowledge and understanding. The Do will always start with a verb. To push students to demonstrate understanding and not only knowledge or skills, be sure to include high-level verbs from Bloom's Taxonomy and Webb's Depth of Knowledge, which will be discussed later in this chapter. Let's look at an example of how a KUD can be developed for both a middle school and an advanced algebra unit.

SEVENTH-GRADE UNIT ON RATIONAL NUMBERS

Sample Common Core standards:

Solve real-world and mathematical problems involving the four operations with rational numbers.

Apply and extend previous understandings of multiplication and division of fractions to multiply and divide rational numbers.

a. *Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as (–1)(–1) = 1 and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.*

b. *Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with nonzero divisor) is a rational number. If p and q are integers, then –(p/q) = (–p)/q = p/(–q). Interpret quotients of rational numbers by describing real-world contexts.*

c. *Apply properties of operations as strategies to multiply and divide rational numbers.*

d. *Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.*

The next example develops a KUD in Figure 3.3 for an Algebra 2 class studying trigonometry.

FIGURE 3.2

SEVENTH-GRADE RATIONAL NUMBERS KUD BASED ON STANDARDS

Students will **K**now . . .	Students will **U**nderstand that . . .	Students will demonstrate knowledge and understanding through the ability to **D**o . . .
K1: New Vocabulary: Rational Number, Convert **K2:** How to add, subtract, multiply, and divide rational numbers. **K3:** The decimal form of a rational number terminates in 0s or eventually repeats. **K4:** How to use division to convert between rational numbers and division.	**U1:** Our numbers follow a pattern that stays the same in all types of numbers. **U2:** A negative in mathematics means "opposite." **U3:** A number can be represented different ways, and there can be many different forms of numbers that are equivalent. **U4:** Only things that are alike can be added or subtracted. **U5:** Rational numbers can be used to represent situations in the real world, which include both positive and negative values. **U6:** Operations with rational numbers follow the same patterns as operations with fractions and integers.	**D1:** Give several examples of how "negative" mean "opposite" in mathematics. **D2:** Convert a rational number to a decimal using long division. **D3:** Explain the different processes for operations with rational numbers. **D4:** Represent real-world contexts with rational numbers. **D5:** Model/demonstrate/illustrate how rational numbers follow the same patterns as integers. (Compare/contrast integers and rational numbers.) **D6:** Explain how the pattern of multiplying negatives results in a positive or negative product and how it relates to "opposite." **D7:** Add, subtract, multiply, and divide rational numbers. **D8:** Convert rational numbers to a decimal equivalent.

ADVANCED ALGEBRA UNIT ON TRIGONOMETRY

Sample Common Core standards:

Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Prove the Pythagorean identity sin2(θ) + cos2(θ) = 1 and use it to find sin(θ), cos(θ), or tan(θ) given sin(θ), cos(θ), or tan(θ) and the quadrant of the angle.

Graph trigonometric functions, showing period, midline, and amplitude.

Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

FIGURE 3.3

ADVANCED ALGEBRA TRIGONOMETRY KUD BASED ON STANDARDS

Students will Know . . .	Students will Understand that . . .	Students will demonstrate knowledge and understanding through the ability to Do . . .
K1: Vocabulary: Unit circle, arc, subtend, radian, period, midline, amplitude, sine, cosine, tangent, maximums, minimums, subtended, periodicity. **K2:** Values of trig functions are interpreted counterclockwise around the unit circle. **K3:** How to find trig values in radical form (similar triangle method) and using technology. **K4:** Shapes and characteristics of the graphs of trig functions. **K5:** Graph functions by hand and with technology for angle from *o* to 2π. **K6:** Radian measure of an angle is the length of the arc on the unit circle subtended by the angle.	**U1:** There are different measurement systems that can be equated, but most have more common uses in specific contexts and areas of study. **U2:** The unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers. **U3:** Trigonometric functions are transformed in the same way as all other functions. **U4:** Trigonometry can be used to model cyclical situations in the real world.	**D1:** Prove Pythagorean identities and use them to find sine, cosine, and tangent values and the quadrant of the angle. **D2:** Graph trigonometric functions showing period, midline, and amplitude. Interpret the key features within a context. **D3:** Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. **D4:** Relate the domain of a function to its graph and to the quantitative relationship it describes. **D5:** Explain how the trigonometric functions in the unit circle relate to all real numbers. **D6:** Explain the impact of transformations on trigonometric functions. Relate the changes to a real-world context or vice versa. **D7:** Explain what a radian is, and convert back and forth between radians and degrees.

WATCH IT!

As you watch Video 3.1, *Planning a Unit Based on Rigorous Mathematical Content*, consider the following questions:

1. How do the teachers make sense of the standards in their units?
2. What is the difference among Knowing, Understanding, and Doing mathematics? How is this shown in unpacking standards?
3. In what ways could the unpacking of the standards influence what the teachers ultimately want students to be able to do and their choice of specific activities in a given lesson?
4. How does writing a KUD for a unit affect assessment?

Video 3.1 Planning a Unit Based on Rigorous Mathematical Content

For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.

As you looked through the understandings in the above units, I hope you thought of other units or topics for which the understandings would also work. For example, in the seventh-grade unit, "**U1:** Our numbers follow a pattern that stays the same in all types of numbers" is true for every unit that explores number's and operations, so it could also be used in integer units in sixth and seventh grades, number systems units in algebra, and even in the advanced algebra unit on trigonometry to connect the unit circle and trigonometric ratios to the real numbers (instead of "**U2:** The unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers"). The given understanding (**U2**) in the advanced algebra unit is a more topical understanding than the **U1** in the seventh-grade unit, which is an overarching understanding. Both types of understandings (topical and overarching) are appropriate and useful for building conceptual understanding and connecting mathematical topics across units and years.

When the same understandings are referred to over and over again, students make the connection with prior knowledge instead of learning the current skill or topic as a *new* thing that they now need to master. This is the power of understanding mathematics. As we learn more and more, the concepts and connections become the building blocks upon which we grow our knowledge and skills. Instead of learning rational numbers as a completely new topic with new procedures to be memorized, students can logically see that this is the same idea as fractions that they have been learning since third grade and integers that they began in sixth grade! Now the connections are made, the same principles and patterns apply, and we just learn new details with a new number group. Math makes sense!

You might have noticed that the KUDs provided were all numbered (K1, K2, etc.). This is not necessary, but it helps significantly when

FIGURE 3.4

WRITING A UNIT KUD

Area	Tips
Know	• The Know category is based on FACTS. These can include vocabulary and definitions, math facts such as trigonometric values, patterns such as integer operation "rules," and so on. • Know statements are straightforward—if you know it, you know it. • Knows are usually written as a list: Bullets are fine. • "How to" do something would be considered a Know. You either know the steps, or you don't.
Understand	• The Understanding category is based on CONCEPTS. They are big connecting ideas. They can be general principles and generalizations. Most often, Understandings provide a "why" to what you are learning. • Understandings have several things you would need to "know" in order to fully be able to understand and explain. Thus, the Know statements should be directly related to one or more Understandings. • Understandings are written in complete sentences and can be preceded by the phrase "Students will understand THAT." For example, "Students will understand quadratic functions" is not an understanding. Quadratic functions are a topic. Instead, "Students will understand *that* quadratic functions follow the patterns and similarities of all families of functions" would be an understanding. • Understandings are revisitable. That is, if a student can come to understand in depth in a single lesson, it is probably not an understanding. • Most units have two to five understandings.

(Continued)

FIGURE 3.4 (Continued)

Do	• The Do category states the mathematical evidence that students should be able to exhibit if they both Know and Understand the content in the unit. It is not where lesson activities are listed—that would be in a specific lesson plan. • Do statements usually begin with verbs, because they describe the actions students should be able to do. • Be sure to include actions that will have students think like the professional—for example, "explain the impact of changing a sample population" is the work of a statistician. • Include higher order verbs to ensure that students are demonstrating understanding (explain, justify, generalize) as well as factual knowledge and skill (define, graph). • Do statements will include the skills of the unit.

referring back to the KUD for planning and for communicating with others. Figure 3.4 suggests some design tips when writing KUDs for your unit.

For most teachers, writing the Understandings for a unit is the most challenging aspect of unpacking standards.

For additional examples of understandings correlated by grade or course, topic, and skill, refer to Figure 3.1 and/or download Additional Understandings in Secondary Mathematics from resources.corwin.com/everymathlearner6-12.

TRY IT! KUD LIST

Purpose: To practice explicitly expressing what students should Know, Understand, and be able to Do as a result of learning based on the standards for the unit.

What are the big ideas or understandings undergirding one of your units? As you look at a unit, what are the K, U, and D? For help and additional ideas for writing understandings, review Figure 3.1. You can also download a unit template to help guide your work.

Most mathematics instructional resources provide the standards being addressed, and many give "essential questions" for each lesson. Some of these are more helpful than others, but in and of themselves, they are often insufficient. For most teachers, developing KUDs for their unit is the most challenging part of differentiation. However, it is worth the effort. The depth at which we come to know our content through this process has many benefits:

- We see connections among mathematics more readily.
- We are ready to answer unexpected questions (see Chapter 4).
- We plan purposeful and targeted lessons (see Chapter 4).
- We recognize conceptual gaps and misconceptions in our students more readily (see Chapter 7).
- We build cohesive units that ensure instruction, tasks, and all forms of assessment reach the desired learning outcomes (see Chapters 4 and 7).
- Differentiation based on anything less may not target essential learning for all learners and thus not have the intended results.

TEACHING UP

One of the misconceptions about differentiation that I have often heard is that differentiation "dummies down" curriculum. Nothing could be further from the truth. Research clearly shows that everyone can learn mathematics at high levels (Boaler, 2015). This is the essence of teaching up. We believe that all students can learn, we can hold all students to high expectations, and we must provide the necessary support for students to accomplish the goal. This is fully developed through clarity of curriculum and expectations, instructional decisions (see Chapter 4), and our classroom culture (see Chapter 5).

The discussion of how mathematics should be taught has been going on for a long time. The National Council of Teachers of Mathematics (NCTM) in 1989 first formally proposed what teaching and learning mathematics might look like through the publication of *Curriculum and Evaluation Standards for School Mathematics*. The Professional Standards for Teaching Mathematics in 1991 and the Assessment Standards for School Mathematics in 1995 followed this. In 2000, NCTM updated these publications with *Principles and Standards for School Mathematics*. These publications and others, including *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) from the National Research Council in 2001, began a serious conversation about what it means to learn mathematics. The learning of mathematics in the mathematics community has never been about memorization and speed. In 2001, the National Research Council in *Adding It Up* (Kilpatrick et al., 2001) suggested five strands for mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. All of these publications and resulting conversations and research have laid the foundation for what teaching and learning in mathematics should be ideally—balancing conceptual understandings and reasoning with procedural skills and strategies, embedded within real-world contexts.

Most recently, we have NCTM's (2014) publication of *Principles to Actions* describing today's mathematics classroom. The first three principles for school mathematics in NCTM's (2014) *Principles to Actions: Ensuring Mathematical Success for All* are as follows:

- Teaching and Learning—Effective teaching should engage students in meaningful learning that stimulates making sense of mathematical ideas and reasoning mathematically.
- Access and Equity—All students have access to a high-quality mathematics curriculum with high expectations and the support and resources to maximize learning potential.
- Curriculum—A curriculum that develops important mathematics along coherent progressions and develops connections among areas of mathematical study and between mathematics and the real world.

So how do we do it? How do we teach and design units and lessons to accomplish all of this? How do we "teach up" to ensure a high-quality mathematics education for all students? The beginning is certainly clarifying the understandings and basing units and instruction around conceptual understandings with embedded skills. While digging into our standards, we need to make sure that we are teaching at or above our grade level or course expectations. Chapter 4 will provide more specific design strategies for helping students to reach appropriate course-level content as well as enrichment and engagement ideas.

Teaching up also involves designing lessons, asking questions, and choosing tasks at a high level of cognitive demand. There are two structures commonly used to determine if we are conducting class at a high level: depth of knowledge (DOK) and cognitive demand.

Both structures have four levels, two levels defined as lower and two defined as upper. DOK, a structure designed by Norman Webb in the late 1990s, was originally designed for mathematics and science standards but has been expanded for all content areas. Cognitive Demand (Smith & Stein, 1998) is a structure specifically for mathematics. Figure 3.5 gives the level names for each framework and their characteristics.

FIGURE 3.5

STRUCTURES FOR COGNITIVE COMPLEXITY

Level	Depth of Knowledge	Cognitive Demand	Description
Lower Level 1	Recall	Memorization	• Reproducing facts, rules, formulas, procedures, or definitions from memory • No connection to concepts
Lower Level 2	Skill/Concept	Procedures Without Connections	• Use information in a familiar situation • Involves two or more steps • Algorithmic. Use a procedure rotely • Very little ambiguity or reasoning involved • No student explanations required
Upper Level 3	Strategic Thinking	Procedures With Connections	• Requires reasoning, developing a plan or a sequence of steps • Some complexity in the task or question • Procedures are to develop connections and conceptual understanding • Multiple representations • Takes cognitive effort
Upper Level 4	Extended Thinking	Doing Mathematics	• Requires an investigation, time to think and process multiple conditions • Requires complex and nonalgorithmic thinking • Explore and understand the nature of mathematical concepts, processes, and relationships • Mathematics of the real world

TRY IT! HOW RIGOROUS IS IT?

Purpose: To practice determining the level of rigor in a given task or problem.

For each of the following tasks, determine the DOK or cognitive demand level. Answers are at the end of the chapter—but don't cheat!

1. Some shoes that I have wanted were just placed on sale! Their original price was $74.99 but now they are on sale for $63.75. I'm wondering if this is a good deal.
   a. What is the percent of decrease from the original price?

      I still didn't have enough money saved up to buy the shoes, so I had to wait. Now the store now has an ad that all sale items have been reduced by one third of the sale price.
   b. What is the new sale price?
   c. What is the overall percent of decrease from the original price?
   d. Explain why your answers are correct. Why can't I just add the original percent decrease and the additional sale decrease to find the overall decrease?
2. Find the next two terms in the arithmetic sequence:

   $\frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$
3. I drove from Phoenix to New York for a total of 2,408 miles. I am hoping to average 55 miles per hour on the trip. I want to drive about 8 hours a day to allow time to stop for lunch and get a good night's sleep. About how many miles per day should I plan to drive and how many days will this take me?
4. Conduct a survey of the freshman class to generate any bivariate data that interest you (for example, number of hours on social media and grade point average). Determine an appropriate sample and how you will randomly sample to gather data. Determine the best display for your raw data (minimum of two) and interpret your data. Consider:

- Do you believe there is a correlation? Causation? How could you know?
- What questions can you ask for further exploration of your data?
- What distributions or statistics might be helpful for interpretation?

5. Use algebra tiles to model simplifying expressions and solving equations. Explain how the tiles model each symbolic step in the process.

   a. $3(x - 1) = 15$ b. $2x + 4 = x - 4$

Most mathematics instruction in the United States occurs at Levels 1 and 2. Level 1 should be a supporting level to enable students to function at Levels 2 and 3. Please understand that these levels are not necessarily hierarchical—that is, you do not move through the levels in order. In fact, starting with a Level 4 task is often a great starting point to create the need to learn the facts, formulas, procedures, and other skills in Levels 1 and 2. Our aim should be for the majority of students' work to be at Levels 2 and 3. In Chapter 4, we will further discuss the selection and design of higher level tasks.

TRY IT! HIGHER AND HIGHER

Purpose: To recognize the complexity with which you most often challenge your students and to push for higher levels.

Make a list of the tasks and questions you have used in class over the past week or two. Write the complexity level (DOK or cognitive demand) next to each one. At what level are most of the tasks and questions with which your students engage? Do you need to make any adjustments? See if you can bump up some of the tasks and questions you have identified as lower level.

There is yet another consideration to teaching up: helping students understand what it means and looks like to be an active learner of mathematics.

WHAT LEARNING MATHEMATICS WITH UNDERSTANDING LOOKS LIKE

How students engage with learning mathematics is equally as important as the content they are learning. In fact, if they are not invested in the process of learning, they may not learn the content to the depth we desire and certainly don't remember it beyond the immediate unit (or sometimes even the next day). As mentioned before, this is not a new way of thinking but has rarely been made an integral part of a standards document or mathematics learning.

Consider It!

What does it look like when students are involved with learning mathematics? What verbs come to mind?

The realization of how important it is that students develop "mathematical habits of mind" has prompted our current focus on describing and expecting that the way students learn mathematics shifts along with the content of what students are learning. Today's standards documents describe student actions for learning in various ways. The Common Core State Standards have described them through the Standards for Mathematical Practice. Other states that have not adopted the Common Core also have process standards that are very similar, and some states that are not using the Common Core State Standards are using the Standards for Mathematical Practice as part of their state's standards document. These behaviors are written as standards to raise the importance of students' actions and thinking in learning mathematics effectively and are not only expected in the classroom but also expected to be assessed as a part of end-of-year testing. Although there are slight differences in the descriptions of each process, the intent and descriptions are consistent from state to state. Figure 3.6 shows alignment among the Standards for Mathematical Practice with other states' process standards. At the time of this writing, other states were looking at amending their standards that may not be reflected in this figure.

The full descriptions of each of these mathematical behaviors can be accessed at the following websites:

- Common Core Standards for Mathematical Practice: http://www.corestandards.org/Math/Practice/
- Nebraska Mathematical Processes: https://www.education.ne.gov/math/Math_Standards/Adopted_2015_Math_Standards/2015_Nebraska_College_and_Career_Standards_for_Mathematics_Vertical.pdf

- Oklahoma Mathematical Actions and Processes: http://sde.ok.gov/sde/sites/ok.gov.sde/files/documents/files/OAS-Math-Final%20Version_2.pdf
- South Carolina Process Standards: https://ed.sc.gov/scdoe/assets/file/agency/scde-grant-opportunities/documents/SCCCRStandardsForMathematicsFinal-PrintOneSide.pdf
- Texas Process Standards: http://www.abileneisd.org/cms/lib2/TX01001461/Centricity/Domain/1943/Texas%20Mathematical%20Process%20Standards%20Aug%202014.pdf
- Virginia Standards of Learning Mathematics Goals: http://www.pen.k12.va.us/testing/sol/standards_docs/mathematics/index.shtml

WATCH IT!

In this video, two teachers discuss the importance of teaching and modeling the eight standards for mathematical practice. The students in the video have been explicitly taught these standards from the beginning of the school year and discuss the role of perseverance (SMP 1) and reasoning (SMP 3) after a group task. As you watch Video 3.2, *Putting the Standards for Mathematical Practice at the Heart of Differentiation,* consider the following questions:

1. How might the teacher have established how students participate in and engage with mathematics content in deep and meaningful ways?
2. What are the pros and cons in the students' actions in learning mathematics with the Standards for Mathematical Practice in mind?
3. How effectively do you explicitly teach and address the practices in your mathematics classroom?
4. Why do you believe there is such an emphasis on mathematical practices (or processes or habits of mind) in the learning of mathematics today?

Video 3.2 Putting the Standards for Mathematical Practice at the Heart of Differentiation

Although the various documents may give different names, there is agreement on what learning mathematics should look like. Combined into a simplified list, here are six aspects of active mathematics learning. Remember that these are describing student actions—not teacher actions! I believe we all do these things as teachers of mathematics. In fact, if we want our students to exhibit

FIGURE 3.6

STANDARDS FOR MATHEMATICAL PRACTICE (SMP) AND OTHER PROCESS STANDARDS

SMP	NE	OK	SC	TX	VA
Make sense of problems and persevere in solving them	Solve mathematical problems	Develop a deep and flexible conceptual understanding	Make sense of problems and persevere in solving them	Use a problem-solving model that incorporates analyzing given information, formulating a plan or strategy, determining a solution, justifying the solution, and evaluating the problem-solving process and the reasonableness of the solution	Mathematical problem solving
Reason abstractly and quantitatively		Develop mathematical reasoning	Reason both contextually and abstractly		Mathematical reasoning
Construct viable arguments and critique the reasoning of others	Communicate mathematical ideas effectively	Develop the ability to communicate mathematically	Use critical thinking skills to justify mathematical reasoning and critique the reasoning of others	Communicate mathematical ideas, reasoning, and their implications using multiple representations, including symbols, diagrams, graphs, and language as appropriate	Mathematical communication
Model with mathematics	Model and represent mathematical problems	Develop the ability to conjecture, model, and generalize	Connect mathematical ideas and real-world situations through modeling	Create and use representations to organize, record, and communicate mathematical ideas	Mathematical representations

SMP	NE	OK	SC	TX	VA
Use appropriate tools strategically		Develop strategies for problem solving	Use a variety of mathematical tools effectively and strategically	Select tools, including real objects, manipulatives, paper and pencil, and technology as appropriate, and techniques, including mental math, estimation, and number sense as appropriate, to solve problems	Mathematical problem solving
Attend to precision		Develop accurate and appropriate procedural fluency	Communicate mathematically and approach mathematical situations with precision	Display, explain, and justify mathematical ideas and arguments using precise mathematical language in written or oral communication	Mathematical problem solving
Look for and make use of structure		Develop the ability to conjecture, model, and generalize	Identify and utilize structure and patterns	Analyze mathematical relationships to connect and communicate mathematical ideas	
Look for and express regularity in repeated reasoning		Develop the ability to conjecture, model, and generalize	Identify and utilize structure and patterns		
	Make mathematical connections	Develop a productive mathematical disposition		Apply mathematics to problems arising in everyday life, society, and the workplace	Mathematical connections

Consider It!

As you read through the combined list of mathematical learning practices, create a plan for how you will teach your students these learning actions and how you will expect and monitor them as your students are engaged with learning mathematics.

these behaviors, we need to model them as well as explicitly teach them to and expect them from our students.

Let's take a closer look at how to implement some of these actions.

Make sense of problems—reason and interpret mathematical situations. This is the obvious beginning, right? But how often do your students barely look at a problem before saying "I don't get it." We often read the problem to the students, interpret the problem for them, and give the first step. It is no wonder that so many students do not know how to make sense of problems for themselves. When students engage in learning mathematics, they wrestle with the context of a problem, what they are looking for, possible ways to start the problem, and multiple solution paths or representations. In addition, when students make sense of problems, they begin to discuss the mathematics they see in a real-world situation and describe a real-world situation that would require the mathematics being learned.

TRY IT! HOW STUDENTS MAKE SENSE OF PROBLEMS

Purpose: To shift "sense-making" to students when facing a new problem

Give the students a rich problem to solve.

1. In partners, have Partner A (student whose birthday is coming next) read the problem and Partner B interpret the problem in his or her own words. This should also include what the solution will look like (for example, ___ feet). Discuss as a class or check in on students' interpretations.
2. Have Partner B suggest a way to start the problem to Partner A. Partner A then suggests another way to start, or agrees to the idea of Partner B, and explains why it will work.
3. Have partners generate strategies to represent and solve the problem.
4. Either together or alone, students go on to solve the problem.

Communicate mathematically—Students explain their thinking mathematically and ask questions of, or build on, other students' explanations. Students who communicate mathematically use correct mathematical vocabulary and multiple representations to communicate their thinking. Beware of accepting only an answer or a retelling of steps as an explanation. Explanations need to include reasoning and the meanings and properties of operations to be considered robust.

TRY IT! STUDENT DISCOURSE

Purpose: To teach healthy mathematical discourse skills

1. Direct mathematical conversations so that they are among the students as much as possible. Do not interpret and redirect questions and answers. Teach students to restate what other students have said. To provide structures for discourse, give students sentence starters such as the following:
   - I agree with ______ because ______
   - Another way to think about this is ______
   - I did it a different way. I ______
   - I disagree with ______ because ______
   - I would like to add on to what ______ said about ______
   - Can you explain what you mean by ______
   - Can you show ______ in another way
   - I think that ______ because ______
2. As the teacher, stay with a student who might be struggling or unsure in his or her communication. Ask questions to help the student clarify his or her thinking rather than move on to another student.
3. Gently correct and provide correct mathematical vocabulary.

Model with mathematics—There are two aspects to modeling mathematics: models of mathematics and how mathematics models the real world. Models of mathematics include using manipulatives such as algebra tiles or models and two-color counters, drawings, tables, graphs, and symbols. Whenever new material is presented in a way that students see relationships, they generate greater brain cell activity and achieve more successful long-term memory storage and retrieval (Willis, 2006, p. 15). We also use mathematics to model the real world. When we solve quantitative problems from the world around us, we are modeling the world with mathematics.

TRY IT! MATHEMATICAL MODELING

Purpose: To make mathematical processes and problems concrete and visual whenever possible

1. It is very important that the concrete or visual model comes *before* the paper-and-pencil process. If the algorithm is taught first, students will not value or want to complete the concrete or visual activity. Also, the concrete or visual task is to develop the conceptual understanding and make sense of the process or algorithm to follow.
2. Connect the concrete or visual explicitly to the skill or process. If it cannot be explicitly connected, it is not a valid model. For example, each step in solving an equation with algebra tiles correlates to the written symbolic step. Once students can correctly solve equations with the manipulative, the process of solving on paper is just recording the process they have already learned with the tiles in mathematical symbols. The role of inverse operations and "unwinding" the order of operations now makes sense instead of being steps to memorize.
3. Challenge students to represent problems in as many different ways as they can through manipulatives and various representations.

Choose and use tools appropriately—Tools can be anything! We usually think of physical tools such as two-color counters, rulers, protractors, and calculators or computers. However, tools can also be mental strategies such as estimation strategies or the distributive property for multiplying a monomial and a trinomial. Knowing which tool or strategy will be appropriate and useful in a given situation is a necessary skill for solving problems and a practical life skill. Often we distribute the tools we will be using in a lesson; for example, today we need compasses. Instead, build a toolbox with all of the mathematical tools in the classroom for table groups or an area where all tools are stored, and challenge students to select the tools they think they will need based on a lesson's description.

TRY IT! THE TOOLBOX

Purpose: Have students select and defend a variety of mathematical tools

1. For a lesson, explain to the students what the task will be and ask them to choose the tools they think they will need.
2. After the task, ask students to reflect on the tools they chose. Did they get what they needed? Why or why not? Did they choose extra tools that were not needed? Why did they select that tool? What have they learned for future selections?

Recognize and use patterns and structures—The more students work with mathematics, the more they can recognize mathematical structures. For example, integers alternate between evens and odds. An even root can only be taken of a positive number to have a real answer, and the root will be positive. An odd root, on the other hand, can be taken of either a positive or negative value, and the root will have the same sign as the radicand. Structure includes understanding why a process works, such as solving a proportion or factoring a trinomial, instead of simply memorizing the steps needed to complete the process. Using a pattern as a mathematical practice at the secondary level is most often related to repeated reasoning rather than finding and using a specific element in a pattern. For example, practicing multiplying exponential factors with the same base repeatedly should lead students to recognize that the product will have the same base and an exponent that is the sum of the exponents in the factors ($x^3 \bullet x^5 = x^8$). Instead of giving students an algorithm, such as the "rules" for adding integers, try modeling thinking about the structures and patterns that are inherent in the operations and use multiple problem examples to find the patterns that lead to the generalized steps.

TRY IT! PATTERN HUNT

Purpose: To make sense of mathematical rules or procedures and recognize mathematical structures that give hints to solutions

1. Give students several problems to solve that use manipulatives, drawings, models, and so on, but not rules or steps. For example, use two-color counters or number lines to model integer addition and subtraction.
2. Generate a list of the problems and their answers from using the models.
3. Have the students find the "shortcuts" or patterns they recognize. For example, after several repetitions and generating a class list of problems and answers, students will see that multiplying two factors with the same sign will have a positive product, but multiplying two factors with opposite signs will have a negative product.
4. This will undoubtedly be the algorithm or rule you wanted to teach, and instead it will be a student discovery.

Attend to precision—Perhaps undergirding all other of these mathematical practices is the ability to attend to precision. Precision includes using correct vocabulary. It includes noticing whether an equation has a plus or minus sign and attending to all other notation. It includes knowing facts and formulas and efficiently using various strategies for operations and problem types. It includes knowing when and how to apply the properties of operations in multiple contexts.

TRY IT! CATCH ME

Purpose: Have students catch you any time you are not mathematically precise

1. Prepare a problem presentation with which you will make several precision errors.
2. Use incorrect or slang vocabulary. Make arithmetic and algebraic mistakes.
3. Tell students in advance that you will be making several errors or imprecise vocabulary, and their job is to find your imprecisions.
4. You can also divide the class into two teams and award points as students collaborate to find errors and imprecisions.

Clarifying content and teaching students how to actively learn mathematics is essential for all mathematics instruction. It is certainly necessary for effective differentiation, which is based on solid curriculum. Just how do rigorous and explicit content, mathematical learning practices, and differentiation fit together?

CONCLUSION

We have developed a complete picture of clarifying content for designing differentiated instruction. We have also discussed the actions students need to employ to learn mathematics with understanding. Effective teaching is not about delivering information or creating meaning. It's melding the two to help students see the meaning in the information they are learning (Sousa & Tomlinson, 2011).

As we prepare to dig into explicitly designing differentiated mathematics instruction, consider the changes in how we teach mathematics as a whole. Figure 3.7 compares the "before and after" of teaching mathematics today, adapted from David Sousa (2015).

FIGURE 3.7

BEFORE AND AFTER OF MATHEMATICAL REASONING

We used to teach mathematics as . . .	But now we teach mathematics as . . .
Problems to be calculated	Situations about which we reason
Procedures to be memorized	Operations that are based on properties with multiple representations and strategies
Isolated topics	Connected concepts
A speed activity for prowess	Problem solving and reasoning for prowess
Teacher led and valued	Student discovered and valued
Something forgettable	Understood, so remembered

There are two keys to differentiation: know your content and know your students. In the previous chapter, we learned how to get to know our students as learners. In this chapter, we looked at knowing our content in depth. In the next chapter, we will look at how knowing our content and our students comes together in powerful differentiation.

FREQUENTLY ASKED QUESTIONS

Q: With the Common Core standards and other state standards so closely aligned, do we really need to go through the work of writing a KUD? Aren't they written somewhere?

A: There are many posts about big ideas online. Some are good and can be a resource. However, some are labeled "conceptual understanding" but are actually fact or skill based. These are not understandings. In addition, there is nothing like the struggle to make sense of the standards to help your own learning and clarify what you want students to come away with. Remember that we want our students to take a challenge and struggle with things that are challenging . . . we need to do the same.

Q: What if my students can't explain their thinking?

A: Chances are pretty good that your students have been asked to tell how they got an answer in mathematics, and this has always been what an "explanation" was. They need to be taught how to construct a mathematical explanation. This can be done by modeling first and foremost, but also ask questions such as "How did you know to do that?" or "What allows you to do that in mathematics (e.g., you can do this by using the distributive property, etc.)?"

Q: What about students who can't reach the standard?

A: First, be very careful of drawing these types of conclusions. Research is showing that almost all students, given time and support, can reach standards. With that said, some students are significantly behind in their mathematics learning. It is very important to teach the grade- or course-level standards in class. When we draw conclusions that certain students are not able to reach the standard and lower the expectations, or worse, lower the instruction level, we widen gaps, not close them. Truthfully, most students can reach the standards given support and, if appropriate, more direct intervention. The Response to Intervention (RTI) structure is designed to help students significantly behind in learning to close gaps and reach as close to grade level as possible, if not actually reach the standard. However, even given the RTI structure, remember that Tier 1 instruction is on grade level through differentiated techniques. Chapter 4 will more specifically address how to design instruction for readiness, as well as interest and learning profile.

Keepsakes and Plans

What are the keepsake ideas from this chapter, those thoughts or ideas that resonated with you that you do not want to forget?

Mathematics Makes Sense:

1.
2.
3.

Themes and Big Ideas in Mathematics:

1.
2.
3.

Teaching Up:

1.
2.
3.

What Learning Mathematics With Understanding Looks Like:

1.
2.
3.

Based on my keepsake ideas, I plan to:

1.
2.

TRY IT ANSWERS

1. DOK 3/procedures with connections. This is a multistep application of computational algorithms. It would be a high DOK 2 if it were not for the explanation requirement.
2. DOK 1/memorization. This is a straightforward algorithm to find a common difference, and the algorithm is probably not needed because the pattern is so simple.
3. DOK 2/procedures without connections. This item is a basic application of computational algorithms. Other than recognizing the need for division, the context does not add depth to understanding the process for solving the problem.
4. DOK 4/doing mathematics. This is an example of what mathematicians do in the real world.
5. DOK 3/procedures with connections. This task uses models to make sense of the algorithms that will be used in simplifying expressions and solving equations prior to formal symbolic instruction.